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FACULTY OF ENGINEERING, CAIRO UNIVERSITY

A DISCRETE FRACTIONAL FOURIER TRANSFORM BASED ON ORTHONORMALIZED McCLELLAN-PARKS EIGENVECTORS

M. T. HANNA[1]

ABSTRACT

A version of the discrete fractional Fourier transform (DFRFT) is developed with the objective of approximating the continuous fractional Fourier transform (FRFT). The eigendecomposition of the discrete Fourier transform (DFT) matrix F represents the main part of the work. First the McClellan-Parks nonorthogonal eigenvectors of F are generated analytically after deriving explicit expressions for the elements of those vectors. Second the Gram-Schmidt technique is applied to orthonormalize the eigenvectors in each eigensubspace individually. Third Hermite-like approximate eigenvectors are generated. Finally exact orthonormal eigenvectors as close as possible to the Hermite-like approximate eigenvectors are obtained by the orthogonal procrustes algorithm. The DFRFT has the properties of unitarity and angle additivity. It approximates its continuous counterpart as demonstrated by the simulation results.

KEYWORDS: Fractional fourier transform, hermite-gaussian functions, orthogonal procrustes algorithm, McClellan-parks eigenvectors.

1. INTRODUCTION

A generalization of the Fourier transform has been recently proposed and termed the fractional Fourier transform (FRFT) in both the mathematics [1,2] and engineering literature [3-5]. Since two successive applications of the Fourier transform result in a time reversed version of the signal, the Fourier transform can be interpreted as a rotation by an angle 0.5π radians in the time-frequency plane. The fractional Fourier transform is a generalization of the Fourier transform because it can be interpreted as a

[1] Associate Professor, Department of Engineering Mathematics and Physics, Cairo University / Fayoum Branch, Fayoum, Egypt.

representation of the signal along an axis making an angle α with the time axis in the time-frequency plane.

Since the discrete Fourier transform (DFT) is the digital counterpart of the Fourier transform, current research activities are taking place in an attempt to develop a digital counterpart of the FRFT to be termed a discrete fractional Fourier transform (DFRFT). At present there is no definitive definition of such a generalization of the DFT. A legitimate definition of the DFRFT should have the properties of unitarity, angle additivity, reduction to the DFT when the angle of rotation α is 0.5π and approximation of its analog counterpart namely the FRFT. Although the first three requirements can be satisfied, the last one can only be approximated. Guided by this goal, a recent definition emerged for the DFRFT that will have eigenvectors that resemble samples of the Hermite-Gaussian functions which are the eigenfunctions of the FRFT [6-7].

A fundamental step in the definition of a DFRFT is the eigendecomposition of the DFT matrix F. In their pioneering work, McClellan and Parks proved that for any order $N \geq 5$, matrix F has only the 4 distinct eigenvalues $\{1, -j, -1, j\}$ and they determined their multiplicities [8]. Moreover they constructed a complete set of linearly independent – but nonorthogonal – eigenvectors. Dickinson and Steiglitz presented a technique for computing orthonormal eigenvectors of matrix F by a detailed analysis of a special matrix S that commutes with F [9]. They proved that if λ is a simple eigenvalue of S then its corresponding eigenvector will also be an eigenvector of F but with a different eigenvalue. Since matrix S is real and symmetric, it has a complete set of real independent eigenvectors that form a basis of the N-dimensional space R^N. Those eigenvectors will be orthogonal if the eigenvalues of S are simple. Based on extensive numerical evidence as well as some analytical results, Dickinson and Steiglitz conjectured that the eigenvalues of S are simple except when N is divisible by 4. In the latter case they proved that S has two zero eigenvalues and they also conjectured that this is the only multiplicity which ever occurs.

Since meeting the requirement of angle additivity of the DFRFT necessitates having orthonormal eigenvectors of matrix F, any development of a DFRFT should start by deciding on a way for getting a complete orthonormal set of eigenvectors of F. In his recent development, Pei et. al. [6] adopted the eigenvectors of the special matrix S. Next they generated samples of the Hermite-Gaussian functions to get approximate eigenvectors of F. Finally they projected them on the eigensubspaces of F in order to get Hermite-like orthonormal eigenvectors of F to be used as a basis for defining a DFRFT that approximates its continuous counterpart.

In an alternative development, Pei et. al. computed Hermite-like orthonormal eigenvectors of F consecutively by solving a series of constrained minimization problems using the Lagrange multipliers method [7]. At each stage, the minimization criterion is the squared norm of the error vector between the exact eigenvector and the approximate eigenvector obtained by sampling the Hermite Gaussian function. Two sets of linear constraints are imposed: the first set is the defining equation of an exact eigenvector and the second set is the requirement that the eigenvector to be evaluated be orthogonal to those evaluated in the previous stages. The QR matrix decomposition technique is applied to the matrix of coefficients of the linear constraints in order to single out a set of linearly independent constraints. The final expression for the exact eigenvector involves a matrix inversion [7].

In the present paper the elegant technique of McClellan and Parks for the analytical generation of independent nonorthogonal eigenvectors of F will be adopted. Next the Gram-Schmidt technique [10] will be applied to orthonormalize these eigenvectors. Finally the Hermite-like approximate eigenvectors will be projected on the last generated orthonormal eigensubspaces using the orthogonal procrustes algorithm [11] in order to get Hermite-like eigenvectors to be used as a basis for defining a DFRFT. This proposed technique has the advantage of avoiding the numerical evaluation of the eigenvectors of matrix S (which are also eigenvectors of F when the eigenvalues of S are simple). It also has the advantage of avoiding the difficulties that might arise when N is divisible by 4.

After introducing the FRFT in section 2, the McClellan-Parks eigendecomposition of matrix F will be explained and explicit expressions will be derived for the elements of the eigenvectors in section 3. The orthonormalization of the McClellan-Parks eigenvectors, the generations of the Hermite-like approximate eigenvectors and the application of the orthogonal procrustes algorithm to get Hermit-like exact eigenvectors will be covered in section 4. The DFRFT will be defined in section 5 and some examples will be given in section 6.

2. THE FRACTIONAL FOURIER TRANSFORM

The fractional Fourier transform (FRFT) is defined by means of the transformation kernel [3]:

$$K_\alpha(t,u)=\begin{cases}\sqrt{\dfrac{1-j\cot\alpha}{2\pi}}\exp\left[j\left(\dfrac{t^2+u^2}{2}\cot\alpha-ut\operatorname{cosec}\alpha\right)\right] & \text{if } \alpha \text{ is not a multiple of } \pi\\ \delta(t-u) & \text{if } \alpha \text{ is a multiple of } 2\pi\\ \delta(t+u) & \text{if } (\alpha+\pi) \text{ is a multiple of } 2\pi\end{cases} \quad (1)$$

where α is the angle of rotation. The transform $X_\alpha(u)$ of the signal x(t) is defined as:

$$X_\alpha(u)=\int_{-\infty}^{\infty}x(t)K_\alpha(t,u)dt. \quad (2)$$

For $\alpha=0$ one gets the identity operator and for $\alpha=0.5\pi$ one gets the classical Fourier transform. The transform kernel has the following property:

$$\int_{-\infty}^{\infty}K_\alpha(t,u)K_\alpha^*(t,u')dt=\delta(u-u'). \quad (3)$$

It means that the kernel functions $K_\alpha(t,u)$ taken as functions of t with parameter u form an orthonormal set. The inverse FRFT is given by:

$$x(t)=\int_{-\infty}^{\infty}X_\alpha(u)K_{-\alpha}(u,t)du. \quad (4)$$

The above formula can be viewed as a way of expressing x(t) on a basis formed by the orthonormal functions $K_{-\alpha}(u,t)$.

The FRFT of the unit impulse $\delta(t)$ is given by (1) and (2) as:

$$FRFT[\delta(t)]=K_\alpha(0,u)=\sqrt{\frac{1-j\cot\alpha}{2\pi}}\exp\left[j\left(\frac{u^2}{2}\cot\alpha\right)\right] \quad \text{if } \alpha \text{ is not a multiple of } \pi. \quad (5)$$

The above function is a chirp, i.e., a complex exponential with a phase that is quadratic in the transform variable u.

Let $H_n(t)$ be the normalized Hermite-Gaussian function of order n defined by:

$$H_n(t) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} h_n(t) \exp\left(-0.5t^2\right) \tag{6}$$

where $h_n(t)$ is the nth order Hermite polynomial defined by [12]:

$$h_n(t) = \sum_{k=0}^{\lfloor 0.5n \rfloor} \frac{(-1)^k n!}{k!(n-2k)!} (2t)^{(n-2k)} \tag{7}$$

with $\lfloor b \rfloor$ being the largest integer not exceeding b. It can be proved that [3]:

$$\int_{-\infty}^{\infty} H_n(t) K_\alpha(t,u) dt = \exp(-jn\alpha) H_n(u). \tag{8}$$

The above equation implies that the Hermite-Gaussian functions $H_n(t)$ are the eigenfunctions of the FRFT operator with the corresponding eigenvalues of $\exp(-jn\alpha)$. Interestingly the eigenfunctions are independent of the angle α although the eigenvalues are dependent on it. Consequently the eigenfunctions of the FRFT are the same as those of the classical Fourier transform (corresponding to $\alpha = 0.5\pi$). The latter has only the 4 distinct eigenvalues $(-j)^k$, $\mathrm{k} = 0,1,2,3$.

3. THE McCLELLAN-PARKS EIGENDECOMPOSITION OF MATRIX F

The discrete Fourier transform (DFT) of the sequence x[n], n = 0, …, N-1 is the sequence X[k], k = 0, …, N-1 defined by [8]:

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi}{N}kn\right) \qquad ,k = 0,\cdots,N-1. \tag{9}$$

The above relation can be compactly expressed as:

$$X = Fx \tag{10}$$

where x and X are the column vectors:

$$x = \begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}^T , \tag{11}$$

$$X = \begin{bmatrix} X[0] & X[1] & \cdots & X[N-1] \end{bmatrix}^T . \tag{12}$$

In this paper the superscripts T, *, + respectively denote the transpose, the complex conjugate and the complex conjugate transpose.

The elements of matrix F in (10) are given by:

$$F_{m,n} = \frac{1}{\sqrt{N}} W^{(m-1)(n-1)} \qquad , m, n = 1, \cdots, N \tag{13}$$

where

$$W = \exp(-j2\pi / N). \tag{14}$$

It is straightforward to show that matrix F is unitary, symmetric but not Hermitian. McClellan and Parks [8] proved that F has only the 4 distinct eigenvalues $\{\pm 1, \pm j\}$ and determined their multiplicities for any N as given by Table 1.

Table 1. The multiplicities of the eigenvalues of matrix F.

N \ λ	1	-j	-1	j
4m	m+1	m	m	m-1
4m+1	m+1	m	m	M
4m+2	m+1	m	m+1	M
4m+3	m+1	m+1	m+1	M

Matrix F has a complete set of orthonormal eigenvectors because it is unitary [13]; however there is no known way for generating such a set analytically. McClellan and Parks contributed an analytical technique for generating a complete set of real linearly independent but nonorthogonal eigenvectors for matrix F. Those eigenvectors are expressed as $z_r = Fu_r \pm u_r$ for $\lambda = \pm 1$ and as $z_r = jFv_r \pm v_r$ for $\lambda = \mp j$ where u_r and v_r are simple column vectors to be given shortly. The values of the index r are given in Table 2 for all possible values of the order N of matrix F. The integer ν appearing in that table is defined by:

$$\nu = \lfloor 0.5N \rfloor + 1 . \tag{15}$$

The simple N-dimensional column vectors u_r for $1 \le r \le \nu$ are defined by:

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \;,\; u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \;,\; u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \;,\; \cdots . \tag{16}$$

Table 2. The McClellan-Parks eigenvectors of matrix F.

N	λ	1	-j	-1	j
	Eigenvectors	$Fu_r + u_r$	$jF\mathrm{v}_\mathrm{r} + \mathrm{v}_\mathrm{r}$	$Fu_r - u_r$	$jF\mathrm{v}_\mathrm{r} - \mathrm{v}_\mathrm{r}$
4m	R	$1,2,\cdots,m,\nu$	$1,\cdots,m-1,\nu-2$	$1,2,\cdots,m$	$1,2,\cdots,m-1$
4m+1	R	$1,2,\cdots,m,\nu$	$1,2,\cdots,m$	$1,2,\cdots,m$	$1,2,\cdots,m$
4m+2	R	$1,2,\cdots,m+1$	$1,2,\cdots,m$	$1,2,\cdots,m+1$	$1,2,\cdots,m$
4m+3	R	$1,2,\cdots,m+1$	$1,2,\cdots,m,\nu-1$	$1,2,\cdots,m+1$	$1,2,\cdots,m$

For $2 \le r \le \nu - 1$, the elements $(u_r)_k$, $k = 1,\cdots,N$ are all zero except for $k = r$, $N + 2 - r$ which are unities. The same applies to $(u_\nu)_k$ when N is odd. When N is even, all elements $(u_\nu)_k$, $k = 1,\cdots,N$ will be zero except for the element $(u_\nu)_\nu$ which will be unity.

On the other hand, the simple N-dimensional column vectors v_r for $1 \le r \le N - \nu$ are defined by:

$$\mathrm{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \quad , \quad \mathrm{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad , \quad \mathrm{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \cdots . \tag{17}$$

All elements $(\mathrm{v}_\mathrm{r})_k$, $k = 1, \cdots, N$ are zero except $(\mathrm{v}_\mathrm{r})_{r+1} = 1$ and $(\mathrm{v}_\mathrm{r})_{N+1-r} = -1$.

In preparation for deriving explicit expressions for the elements of the McClellan-Parks eigenvectors, one starts by defining the vectors:

$$x_r = F u_r \qquad , 1 \le r \le \nu \tag{18}$$

$$y_r = jF\mathrm{v}_\mathrm{r} \qquad , 1 \le r \le N - \nu . \tag{19}$$

From definition (13) of matrix F and definition (16) of vector u_1, one gets the elements of vectors x_1 as:

$$(x_1)_k = 1/\sqrt{N} \qquad , k = 1, \cdots, N . \tag{20}$$

For $2 \le r \le \nu - 1$, one gets:

$$\begin{aligned} (x_r)_k &= \frac{1}{\sqrt{N}} \left(W^{(k-1)(r-1)} + W^{(k-1)(N+1-r)} \right) \\ &= \frac{1}{\sqrt{N}} \left(W^{(k-1)(r-1)} + W^{-(k-1)(r-1)} \right) \quad , 1 \le k \le N \quad , \quad 2 \le r \le \nu - 1 . \\ &= \frac{2}{\sqrt{N}} \cos\left(\frac{2\pi}{N} (k-1)(r-1) \right) \end{aligned} \tag{21}$$

For the last vector x_ν in the set of (18) two cases arise depending on N being odd or even. For odd N, the elements of x_ν are also given by (21). For even N, vector u_ν has only one nonzero element, and consequently the elements of x_ν reduce to:

$$(x_\nu)_k = \frac{1}{\sqrt{N}} W^{(k-1)(\nu-1)} = \frac{1}{\sqrt{N}} W^{0.5N(k-1)} = \frac{1}{\sqrt{N}} (-1)^{(k-1)} \quad , 1 \le k \le N \quad , \mathrm{N}\,\mathrm{even} . \tag{22}$$

On the other hand, the elements of vector y_r of (19) can be derived as:

$$\begin{aligned}(y_r)_k &= j\left(F_{k,r+1} - F_{k,N+1-r}\right) \\ &= j\frac{1}{\sqrt{N}}\left(W^{(k-1)r} - W^{(k-1)(N-r)}\right) \\ &= j\frac{1}{\sqrt{N}}\left(W^{(k-1)r} - W^{-(k-1)r}\right) \qquad , 1 \le k \le N \ , \ 1 \le r \le N-\nu . \\ &= \frac{2}{\sqrt{N}}\sin\left(\frac{2\pi}{N}(k-1)r\right)\end{aligned} \tag{23}$$

By utilizing (20)-(23), one can derive explicit expressions for the elements of the McClellan-Parks eigenvectors of Table 2. Those expressions are given in Tables 3 and 4 corresponding to the eigenvalues $\lambda = \mp j$ and $\lambda = \pm 1$ respectively.

Table 3. The elements $(z_r)_k$, $\mathrm{k} = 1, \cdots, \mathrm{N}$ of the eigenvectors $z_r = jF\mathrm{v_r} \pm \mathrm{v_r}$

λ	-j	j
Eigenvectors	$z_r = jF\mathrm{v_r} + \mathrm{v_r}$	$z_r = jF\mathrm{v_r} - \mathrm{v_r}$
$1 \le r \le N-\nu$	$(z_r)_k = \begin{cases} \frac{2}{\sqrt{N}}\sin\left(\frac{2\pi}{N}(k-1)r\right)+1 & ,k=r+1 \\ \frac{2}{\sqrt{N}}\sin\left(\frac{2\pi}{N}(k-1)r\right)-1 & ,k=N+1-r \\ \frac{2}{\sqrt{N}}\sin\left(\frac{2\pi}{N}(k-1)r\right) & \text{otherwise} \end{cases}$	$(z_r)_k = \begin{cases} \frac{2}{\sqrt{N}}\sin\left(\frac{2\pi}{N}(k-1)r\right)-1 & ,k=r+1 \\ \frac{2}{\sqrt{N}}\sin\left(\frac{2\pi}{N}(k-1)r\right)+1 & ,k=N+1-r \\ \frac{2}{\sqrt{N}}\sin\left(\frac{2\pi}{N}(k-1)r\right) & \text{otherwise} \end{cases}$

4. ORTHONORMAL HERMITE-LIKE EIGENVECTORS OF MATRIX F

The analytically generated nonorthogonal McClellan-Parks eigenvectors will be first orthonormalized by the Gram-Schmidt technique in order to get orthonormal basis for the 4 eigensubspaces corresponding to the 4 distinct eigenvalues of F. Next Hermite-like approximate eigenvectors will be generated. Finally Hermite-like exact eigenvectors will be obtained.

4.1 Orthonormalizing the McClellan-Parks Eigenvectors

Since matrix F is unitary, eigenvectors corresponding to distinct eigenvalues are orthogonal. Since F has only 4 distinct eigenvalues, the N-dimensional real space R^N will be divided into 4 eigensubspaces E_k corresponding to the 4 distinct eigenvalues $\lambda_k = (-j)^k$, $k = 0,1,2,3$. The McClellan-Parks technique of last section generates real eigenvectors. For each eigenvalue λ_k, the corresponding nonorthogonal eigenvectors span the eigensubspace E_k. Since eigenvectors lying in different spaces E_k are orthogonal, it remains to apply the Gram-Schmidt technique to the McClellan-Parks eigenvectors in each subspace E_k individually in order to get orthonormal eigenvectors. The resulting orthonormal eigenvectors of E_k will be arranged to form the columns of a matrix to be denoted by V_k.

The approach of the present paper is distinct from that of [6] in generating the 4 matrices $V_k, k = 0,\cdots,3$ by first analytically generating the McClellan-Parks eigenvectors and second applying the Gram-Schmidt technique to the individual eigensubspaces. In [6], Pei et. al. utilized the result of Dickinson and Steiglitz [9] that the eigenvectors of F are the same as those of a special simple matrix S under certain assumptions. The approach of the present paper is less computationally demanding than that of [6] which necessitates the numerical evaluation of all eigenvectors of S. It has the merit of not needing to revert to a classification technique for assigning the eigenvectors of S to the 4 eigensubspaces of F. It has the extra merit of avoiding the difficulties that arise when S has a repeated eigenvalue. Actually in the latter case an ordinary eigenvector evaluation program will not generate the circularly even and odd eigenvectors corresponding to a double eigenvalue of S and one has to take into account some computational considerations in order to generate the right circularly symmetric eigenvectors of F [9].

4.2 Generation of Hermite-Like Approximate Eigenvectors

Since one of the goals in developing a discrete fractional Fourier transform is to approximate the continuous fractional Fourier transform, one starts by generating samples of the Hermite-Gaussian functions $H_n(t)$ of (6) which are the eigenfunctions of the FRFT. More specifically, one starts by generating the samples:

$$\phi_n[k] = h_n\left(k\sqrt{2\pi / N}\right)\exp\left(-k^2\pi / N\right) . \tag{24}$$

Next a sequence $\overline{\phi_n}[k]$ is defined in the range [0, N-1] by shifting the sample $\phi_n[k]$ in the following way:

$$\overline{\phi_n}[k] = \begin{cases} \phi_n[k] & \text{for } 0 \le k \le \lceil 0.5N \rceil - 1 \\ \phi_n[k-N] & \text{for } \lceil 0.5N \rceil \le \mathrm{k} \le \mathrm{N-1} \end{cases} \tag{25}$$

where $\lceil b \rceil$ is the smallest integer larger than or equal to b. The discrete Fourier transform (DFT) of the sequence $\overline{\phi_n}[k]$ is defined by:

$$DFT\left\{\overline{\phi_n}[k]\right\} = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\overline{\phi_n}[k]\exp\left(-j2\pi km / N\right) \qquad , m = 0, \cdots, N-1. \tag{26}$$

The following approximate formula has been recently derived [6,7]:

$$DFT\left\{\overline{\phi_n}[k]\right\} \approx (-j)^{\mathrm{n}}\,\overline{\phi_n}[m] \qquad , m = 0, \cdots, N-1 . \tag{27}$$

The above formula implies that the sequence $\overline{\phi_n}[k], k = 0, \cdots, N-1$is an approximate eigensequence of the DFT operator with a corresponding eigenvalue $(-j)^n$. The approximation error grows with the order n of the Hermite-Gaussian function $H_n(t)$ [6].

Define the N-dimensional vector w_n as:

$$w_n = \left[\overline{\phi_n}[0] \quad \overline{\phi_n}[1] \quad \cdots \quad \overline{\phi_n}[N-1]\right]^T \tag{28}$$

and the normalized vector u_n as:

$$u_n = w_n / \|w_n\| . \tag{29}$$

Following the matrix representation (10) of the DFT operator, Eqs. (27)-(29) imply that:

$$Fu_n \approx (-j)^n u_n \ . \tag{30}$$

The above formula indicates that u_n is an approximate eigenvector of matrix F with the corresponding eigenvalue $(-j)^n$. This holds for any nonnegative integer n since the Hermite polynomials and the Hermite-Gaussian functions are defined for all nonnegative integers n. Fortunately the eigenvalues have emerged exactly since matrix F has only the 4 distinct eigenvalues $(-j)^k, k=0,\cdots,3$ as it was proved in [8]. Since the unitary matrix F of order N has only N linearly independent eigenvectors, one should select a set of indices $\Psi=\{n_1,n_2,\cdots,n_N\}$ so that the corresponding vectors $u_{n_k}, k=1,\cdots,N$ will be adopted as the approximate Hermite-like eigenvectors of F. Those indices should be selected such that the eigenvalues $(-j)^{n_k}, k=1,\cdots,N$ will satisfy the multiplicities requirement of Table 1. Moreover, those indices should be as small as possible in order to reduce the approximation error as explained before. Therefore the set Ψ should be selected as given in Table 5 suggested in [6]. A careful examination of this table reveals that for odd N, the set Ψ is $\{0,1,\cdots,N-1\}$; and for even N, the set Ψ is $\{0,1,\cdots,N-2,N\}$.

Table 5. The set of the indices $\Psi=\{n_1,n_2,\cdots,n_N\}$.

N	$n_1,n_2,\cdots,n_N$
4m	$0,1,2,\cdots,(4m-2),4m$
4m+1	$0,1,2,\cdots,(4m-1),4m$
4m+2	$0,1,2,\cdots,4m,(4m+2)$
4m+3	$0,1,2,\cdots,(4m+1),(4m+2)$

The Hermite-like approximate eigenvectors $u_{n_k}, k=1,\cdots,N$ will be classified into 4 sets corresponding to the 4 distinct eigenvalues $\lambda_k=(-j)^k, k=0,\cdots,3$ of F. The vectors corresponding to λ_k will form the columns of a matrix to be denoted by U_k.

4.3 Hermite-Like Orthonormal Exact Eigenvectors

The columns of matrix V_k introduced in Sec 4.1 are orthonormal eigenvectors of F that span the eigensubspace corresponding to the eigenvalue $(-j)^k$. However those eigenvectors are not Hermite-like and consequently a DFRFT based on them will not be a good approximation for the FRFT. On the other hand, although the columns of matrix U_k defined in Sec 4.2 are Hermite-like, they are approximate rather than exact eigenvectors of F. Therefore one should search for eigenvectors of F that are as close as possible to the Hermite-like approximate eigenvectors. One is naturally led to using the orthogonal procrustes algorithm [11] to be explained next.

Let $\hat{U}_k$ be the matrix of eigenvectors of F corresponding to the eigenvalue $(-j)^k$ that is as close as possible to matrix U_k. The matrix $\hat{U}_k$ will be expressed as:

$$\hat{U}_k = V_k Q_k \tag{31}$$

where Q_k is a unitary matrix. Since the columns of $\hat{U}_k$ are linear combinations of those of V_k, they are exact eigenvectors corresponding to the same eigenvalue $(-j)^k$. Moreover they are orthonormal because the columns of V_k are orthonormal and matrix Q_k is unitary. In order for them to be as close as possible to the Hermite-like columns of U_k, one should minimize the Frobenius norm:

$$J_k = \left\| U_k - \hat{U}_k \right\|_F = \left\| U_k - V_k Q_k \right\|_F \ . \tag{32}$$

The unitary matrix Q_k, which minimizes the above criterion, has been obtained as follows [11]:

a) Form the square matrix C_k:

$$C_k = V_k^+ U_k \tag{33}$$

b) Obtain the singular value decomposition of C_k:

$$C_k = A_k \Sigma_k B_k^+ \tag{34}$$

c) Compute matrix Q_k:

$$Q_k = A_k B_k^+ \ . \tag{35}$$

One should mention that although the orthogonal procrustes algorithm was employed in [6], it was erroneously applied since $\hat{U}_k$ was wrongly taken as $\hat{U}_k = Q_k V_k$ rather than as $\hat{U}_k = V_k Q_k$.

Upon assessing the computational complexity one notices that the method of the present paper requires the application of the Gram-Schmidt method once and the singular value decomposition four times. This is to be compared with the method of Pei et. al. [7] which requires the application of both the QR decomposition technique and matrix inversion N times.

5. A DISCRETE FRACTIONAL FOURIER TRANSFORM

The Hermite-like orthonormal eigenvectors of matrix F obtained in Section 4 will be arranged to form the unitary matrix $\hat{U}$ of order N as follows:

$$\hat{U} = \begin{bmatrix} \hat{u}_{n_1} & \hat{u}_{n_2} & \cdots & \hat{u}_{n_N} \end{bmatrix} \tag{36}$$

where the exact eigenvector $\hat{u}_n$ corresponds to the approximate eigenvector u_n of (30). The set of indices $\{n_1, \cdots, n_N\}$ is given in Table 5. Consequently matrix F has the following eigendecomposition:

$$F = \hat{U} D \hat{U}^{+} \tag{37}$$

where D is a diagonal matrix defined by its diagonal elements as:

$$D = diag\{(-j)^{n_1}, \cdots, (-j)^{n_N}\}. \tag{38}$$

The transform kernel of the discrete fractional Fourier transform (DFRFT) is defined by [6]:

$$F^{\frac{2}{\pi}\alpha} = \hat{U} D^{\frac{2}{\pi}\alpha} \hat{U}^{+} \tag{39}$$

where the diagonal matrix $D^{\frac{2}{\pi}\alpha}$ is defined by:

$$D^{\frac{2}{\pi}\alpha} = diag\{\exp(-j\alpha n_1), \cdots, \exp(-j\alpha n_N)\}. \tag{40}$$

When the angle of rotation $\alpha = 0.5\pi$, the transform kernel of (39) reduces to matrix F of the DFT. On the other hand when $\alpha = 0$, the kernel reduces to the identity matrix.

Let the column vector x represent the time-domain signal as defined in (11). The discrete fractional Fourier transform X_α of x is defined by:

$$X_\alpha = F^{\frac{2}{\pi}\alpha} x \ . \tag{41}$$

Combining (41) and (39), one obtains:

$$X_\alpha = \hat{U} D^{\frac{2}{\pi}\alpha} a = \sum_{k=1}^{N} \exp(-j\alpha n_k) a_k \hat{u}_{n_k} \tag{42}$$

where

$$a = \hat{U}^{+} x = [a_1 \ \cdots \ a_N]^T \ . \tag{43}$$

One notices that matrix $\hat{U}$ and vector a are independent of the angle of rotation α and consequently they are computed only once.

6. EXAMPLES

The DFRFT of the discrete-time impulse $\delta[n]$ is computed using N = 35 and for $\alpha = 1, 0.475\pi, 0.49\pi$ and 0.5π radians. The results are shown in Fig. 1 where the solid line represents the real part and the dashed line represents the imaginary part and where the horizontal axis represents the discrete time index (n). For $\alpha = 0.5\pi$ the DFRFT reduces to the classical discrete Fourier transform (DFT) and from (9) one finds that the DFT of $\delta[n]$ is simply $X[k] = 1/\sqrt{N}$ in agreement with Fig. 1(d). Actually as α approaches 0.5π, the DFRFT approaches the DFT as can be seen from Figs. 1(b,c,d).

Since the DFRFT has been developed with the objective of approximating its continuous counterpart, it is appropriate to plot the FRFT of the continuous-time impulse $\delta(t)$ given by (5) for the same values of α of Fig. 1. When $\alpha = 0.5\pi$, the FRFT reduces to the classical Fourier transform and consequently the FRFT of $\delta(t)$ will be $1/\sqrt{2\pi}$ as can be seen from (5). For the sake of comparison, the FRFT will be scaled by the factor $\sqrt{2\pi / N}$ so that it will coincide with the DFRFT when $\alpha = 0.5\pi$ and the signal is the unit impulse. Fig. 2 is the continuous counterpart of Fig. 1 where the frequency variable u in (5) has been sampled with an increment of $\sqrt{2\pi / N}$. As α approaches the value 0.5π, the scaled FRFT approaches the real constant value $1/\sqrt{N}$. The similarity between Figs. 1 and 2 demonstrates clearly that the DFRFT approximates the continuous FRFT.

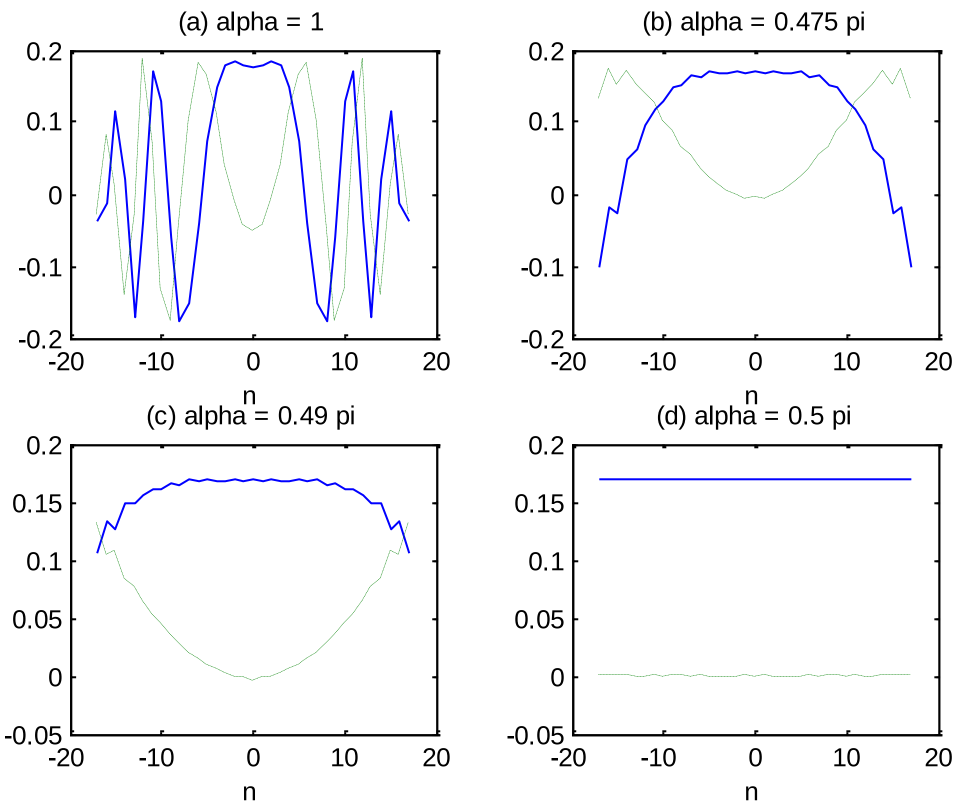

Fig. 1. The DFRFT of a discrete-time impulse.

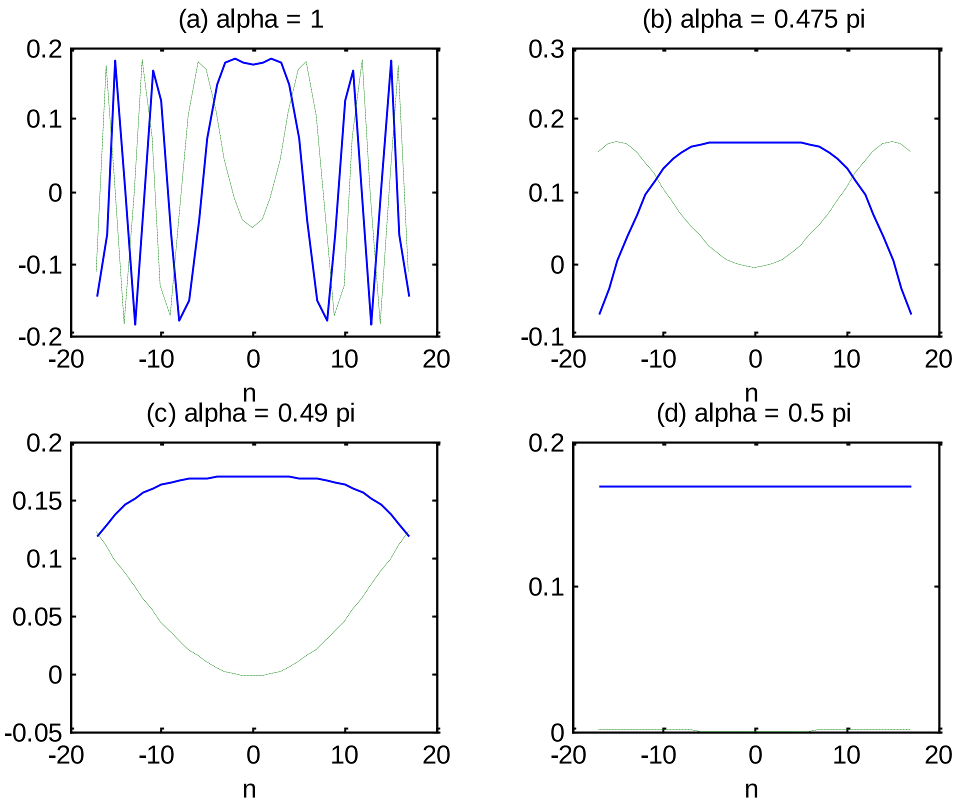

Fig. 2. The (scaled) FRFT of a continuous-time impulse.

7. CONCLUSION

A version of the discrete fractional Fourier transform (DFRFT) has been developed to approximate its continuous counterpart, namely the fractional Fourier transform (FRFT). A major part of the work has been the eigendecomposition of the discrete Fourier transform (DFT) matrix F. In previous work, orthonormal eigenvectors of F were obtained by applying the numerical eigendecomposition techniques to a special matrix S having the same eigenvectors as matrix F under certain assumptions. In the present paper, nonorthogonal eigenvectors have been first generated by the McClellan-Parks method after deriving explicit expressions for the elements of those eigenvectors. Second the Gram-Schmidt technique has been applied to orthonormalize the eigenvectors in each eigensubspace individually since eigenvectors corresponding to distinct eigenvalues are orthogonal by the unitarity of matrix F. In order to approximate the continuous FRFT, Hermite-like approximate eigenvectors are obtained by sampling the Hermite-Gaussian eigenfunctions of the continuous Fourier transform. Finally exact orthonormal eigenvectors as close as possible to the Hermite-like approximate eigenvectors are obtained by the orthogonal procrustes algorithm.

REFERENCES

1. Namias, V., "The fractional order Fourier transform and its application to quantum mechanics," J. Inst. Math. Appl., vol. 25, pp. 241-265, 1980.
2. McBride, A.C. and Kerr, F.H., "On Namias' fractional Fourier transforms," IMA J. Appl. Math., vol. 39, pp. 159-175, 1987.
3. Almeida, L.B., "The fractional Fourier transform and time-frequency representations," IEEE transactions on Signal Processing, vol. SP-42, pp. 3084-3091, Nov 1994.
4. Ozaktas, H.M., Barshan, B., Mendlovic, D. and Onural, L., "Convolution, filtering, and multiplexing in fractional Fourier domains and their relationship to chirp and wavelet transforms," Journal of the Optical Society of America A, vol. 11, pp. 547-559, February 1994.
5. Lohmann, A.W., "Image rotation, Wigner rotation and fractional Fourier transform," Journal of the Optical Society of America A, vol. 10, pp. 2181-2186, 1993.
6. Pei, S.-C., Yeh, M.-H. and Tseng, C.-C., "Discrete fractional Fourier transform based on orthogonal projections," IEEE transactions on Signal Processing, vol. SP-47, pp. 1335-1348, May 1999.
7. Pei, S.-C., Tseng, C.-C. and Yeh, M.-H., "A new discrete fractional Fourier transform based on constrained eigendecomposition of DFT matrix by Lagrange multiplier method," IEEE Transactions on Circuits and Systems, Part II, vol. 46, pp. 1240-1245, September 1999.
8. McClellan, J.H. and Parks, T.W., "Eigenvalue and eigenvector decomposition of the discrete Fourier transform," IEEE Transactions on Audio and Electroacoustics, vol. AU-20, pp. 66-74, March 1972.
9. Dickinson, B.W. and Steiglitz, K., "Eigenvectors and functions of the discrete Fourier transform," IEEE Transactions on Acoustics, Speech and Signal Processing, vol. ASSP-30, pp. 25-31, February 1982.
10. Friedberg, S.H., Insel, A.J. and Spence, L.E., Linear Algebra, Prentice-Hall, Englewood Cliffs, N.J., 1989.
11. Golub, G.H. and Van Loan, C.F., Matrix Computations, Johns Hopkins University Press, Baltimore, M.D., 1989.
12. Andrews, L.C., Special Functions of Mathematics for Engineers, 2nd Edition, SPIE Press, Bellingham, WA and Oxford University Press, Oxford, U.K., 1998.
13. Gantmacher, F.R., The Theory of Matrices, vol. 1, 2nd ed., Chelsea, New York, 1990.

تحويل فورير الكسرى المتقطع المبنى على المتجهات الذاتية لمكللان وبارك

يهدف البحث إلى الوصول إلى تحويل فورير كسرى متقطع يتميز بأنه تقريب لتحويل فورير الكسرى المستمر. ويمثل التحليل الذاتى لمصفوفة تحويل فورير المتقطع الجزء الرئيسى من هذا البحث، فأولا تم توليد المتجهات الذاتية غير المتعامدة لمكللان وبارك بطريقة تحليلية وذلك بعد اشتقاق تعبيرات صريحة لعناصر هذه المتجهات، وثانيا تم تطبيق طريقة جرام-شميت للحصول على متجهات ذاتية متعامدة فى كل فراغ ذاتى جزئى على حدة، وثالثا تم توليد متجهات ذاتية تقريبية تشبه دوال هرميت وأخيرا تم الحصول على متجهات ذاتية متعامدة دقيقة وقريبة من المتجهات الذاتية التقريبية الشبيهة بدوال هرميت وذلك باستخدام أسلوب التوفيق القسرى المتعامد.